MATH2050C Selected Solution to Assignment 1

Section 2.2.

(8b) Solve $2x - 1 = |x - 5|$.

Solution Consider two cases: (a) $x < 5$ and (b) $x \geq 5$. In (a) the equation becomes $2x - 1 = -(x - 5)$. Solve it to get $x = 2$. In (b) the equation becomes $2x - 1 = x - 5$ which is solved to get $x = -4$. Conclusion: $x = 2, -4$ are the solutions for this equation.

Note: You may consider (a) $x \leq 5$ and (b) $x > 5$ as well. There is no essential difference.

(10b) Solve $|x| + |x + 1| < 2$.

Solution Consider three cases: (a) $x < -1$, (b) $x \in [-1, 0]$, and (c) $x > 0$. In (a) the inequality becomes $-x - (x+1) < 2$ which is solved to get $x > -3/2$. Hence the solution is $(-3/2, -1)$. In (b), the inequality becomes $-x + (x + 1) < 2$ which always holds. Hence the solution is $[-1, 0]$. In (c), the inequality becomes $x + (x + 1) < 2$ which is solved to get $x < 1/2$. Putting together, the solution of this inequality is $(-3/2, 1/2)$.

(14b) Determine and sketch $\{(x, y) : \ |x| + |y| = 1\}$.

Solution The figure is the rhombus with vertices at $(1, 0), (0, 1), (-1, 0), (0, -1)$.

(17) Show that for distinct a, b, there exist ε-n'd U of a and V of b such that $U \cap V = \phi$.

Solution Assume $b > a$. Letting $r = (b-a)/2$, $U = V_r(a)$ and $V = V_r(b)$ satisfy our requirement. Recall that $V_r(a) = (a - r, a + r)$.

Supplementary Problems

The following optional problems are for you to practise mathematical induction.

1. Prove Bernoulli's Inequality:

$$(1 + x)^n \geq 1 + nx, \quad x \geq -1, n \geq 1 \ .$$

   Solution See 2.1 Text.

2. Prove Binomial theorem: For real a, b,

$$(a + b)^n = \sum_{k=0}^{n} C_k^n a^{n-k} b^k \ , \quad n \geq 1 \ .$$

   Here $C_k^n = \frac{n!}{k!(n-k)!}$ and $0! = 1$.

Solution Use MI. It is obvious when $n = 1$. Now assume n is true. Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n} C_k^n a^{n-k} b^k \quad \text{by induction hypothesis} \\
&= \sum_{k=0}^{n} C_k^n a^{n-k+1} b^k + \sum_{k=0}^{n} C_k^n a^{n-k} b^{k+1} \\
&= \sum_{k=1}^{n} (C_k^n + C_{k-1}^n) a^{n-k+1} b^k + a^{n+1} + b^{n+1} \\
&= \sum_{k=1}^{n} C_k^{n+1} a^{n-k+1} b^k + a^{n+1} + b^{n+1} \\
&= \sum_{k=0}^{n+1} C_k^{n+1} a_k^{n+1-k} b^k \ .
\end{aligned}$$

The formula for all n by induction.

3. Prove the GM-AM Inequality: For non-negative $a_1, a_2, \cdots, a_n$,

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \ , \quad n \ge 1,$$

and equality in the inequality holds iff all a_j's are equal.

Solution First show it is true for $n = 2^k, k \ge 1$. When $k = 1$, the inequality becomes

$$\frac{1}{2}(a+b) \ge ab, \ a, b \ge 0,$$

and equality holds iff $a = b$. This comes from the relation $(x-y)^2 > 0$ whenever $x \ne y$ (taking $a = \sqrt{x}$ and $b = \sqrt{y}$). Now assume the case $n = 2^k$ is true. We have

$$\begin{aligned}
a_1 + \cdots + a_{2^{k+1}} &= (a_1 + \cdots + a_{2^k}) + (a_{2^k+1} + \cdots + a_{2^{k+1}}) \\
&\ge 2\left[(a_1 + \cdots + a_{2^k})(a_{2^k+1} + \cdots + a_{2^{k+1}})\right]^{1/2} \\
&\ge 2\left[2^k (a_1 \cdots a_{2^k})^{1/2^k} \times 2^k (a_{2^k+1} \cdots a_{2^{k+1}})^{1/2^k}\right]^{1/2}) \quad \text{(induction hypothesis)} \\
&= 2^{k+1} (a_1 \cdots a_{2k+1})^{1/2^{k+1}} \ .
\end{aligned}$$

Also, equality holds iff all a_j's are equal. Now, for a general n. We fix some k such that $n < 2^k$ and consider $a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2^k}$ where $a_{n+1} = \cdots = a_{2^k} = (a_1 + \cdots + a_n)/n$. Plugging this in the inequality for 2^k, after some computations, yields the inequality for n. Also equality holds iff all a_j's are equal.